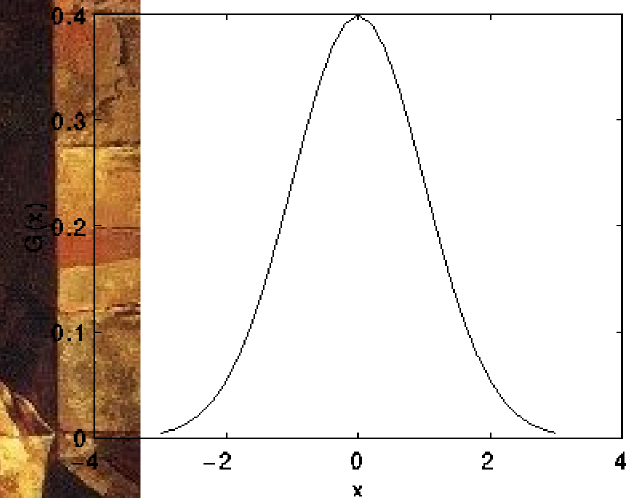
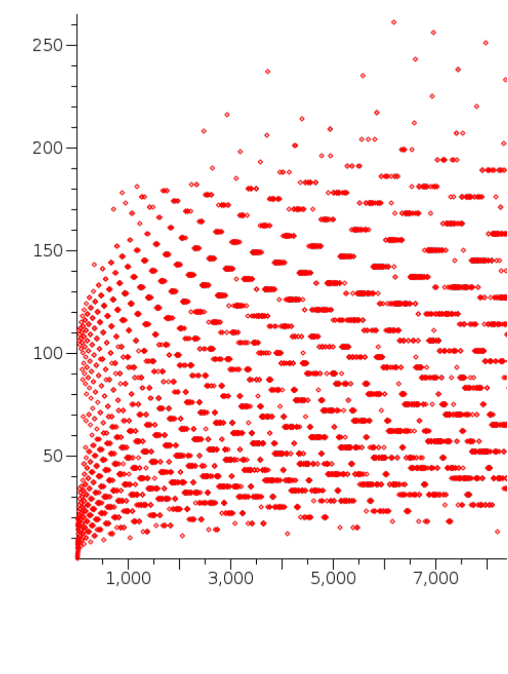
OF TRIANGLES, GAS, PRICE, AND MEN

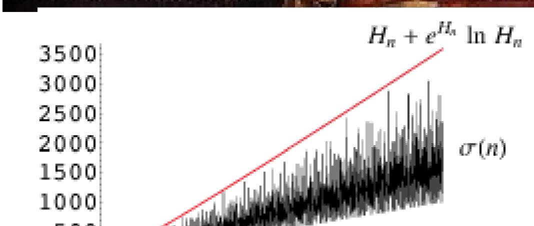
Cédric Villani
Univ. de Lyon & Institut Henri Poincaré

« Mathematics in a complex world »
Milano, March 1, 2013

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$







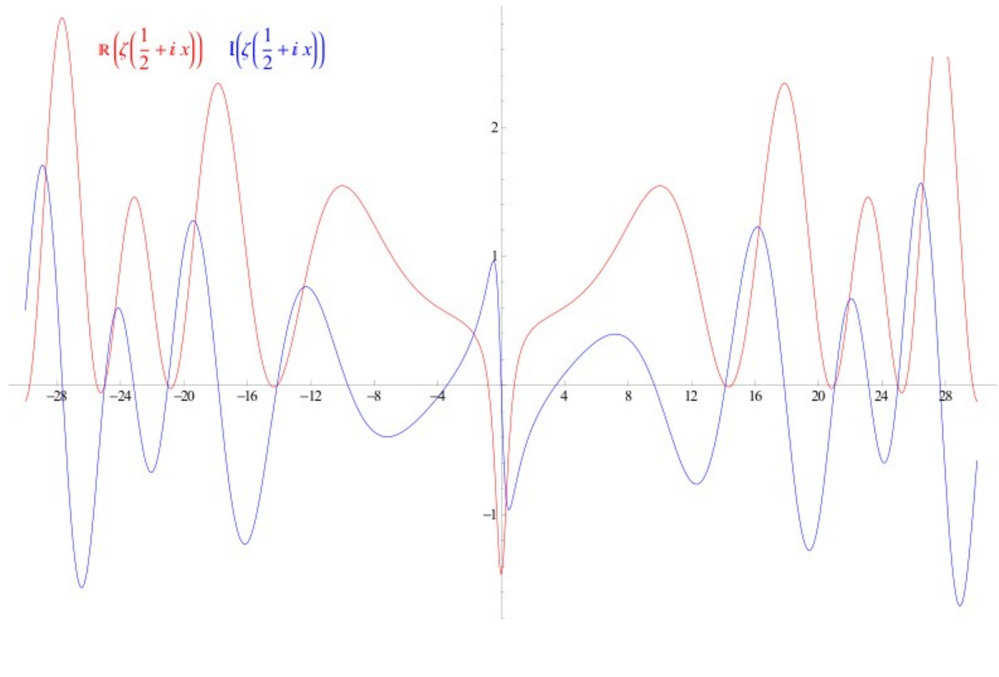
$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0$$

Riemann Hypothesis

(deepest scientific mystery of our times?)

Bernhard Riemann 1826-1866

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

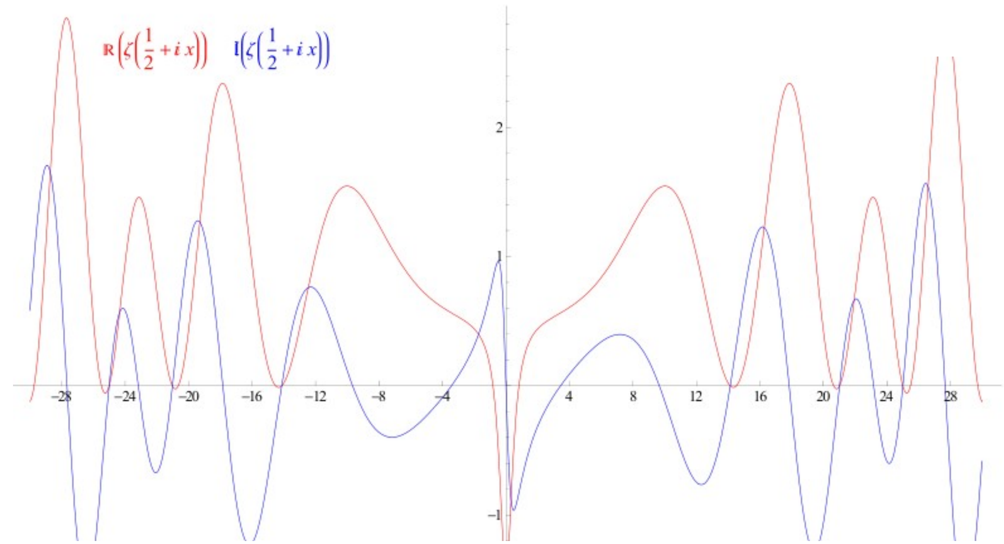


Riemann Hypothesis

(deepest scientific mystery of our times?)

Bernhard Riemann 1826-1866

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$



theguardian

Printing sponsored by:
Kodak
All-in-One Printers

Maths holy grail could bring disaster for internet

Two of the seven million dollar challenges that have baffled for more than a century may be close to being solved

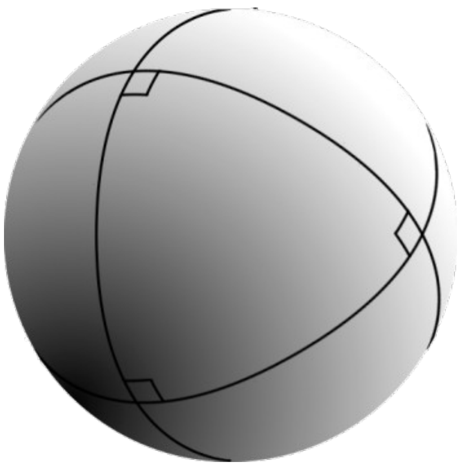
Tim Radford, science editor
The Guardian, Tuesday 7 September 2004

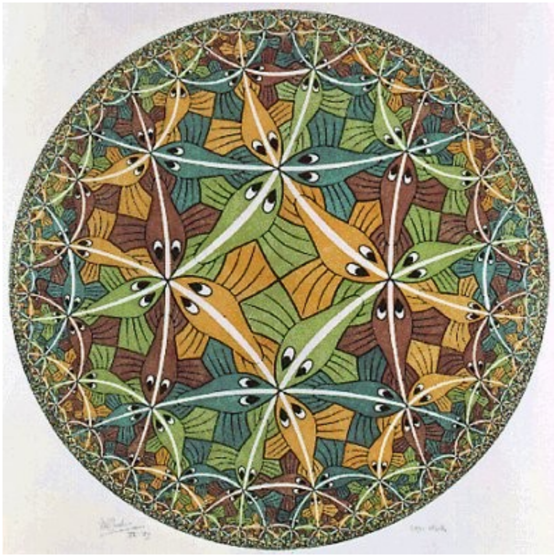
Riemannian (= non-Euclidean) geometry

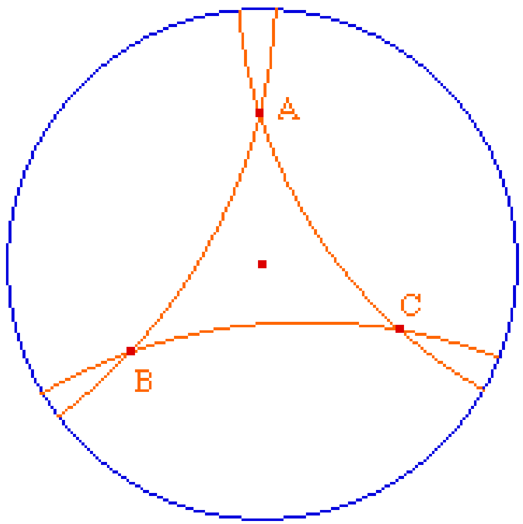
At each location, the units of length and angles may change

Shortest path (= geodesics) are curved!!

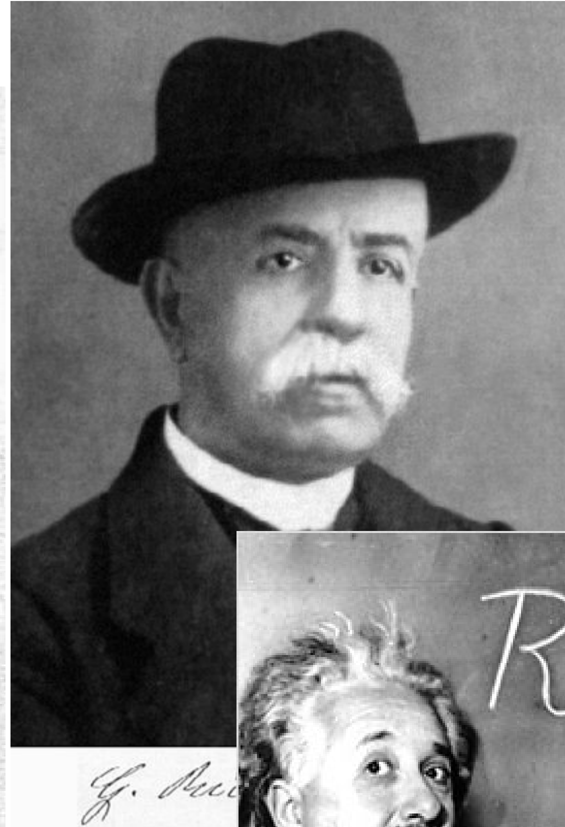
Geodesics can tend to get closer (positive curvature, fat triangles)
or to get further apart (negative curvature, skinny triangles)

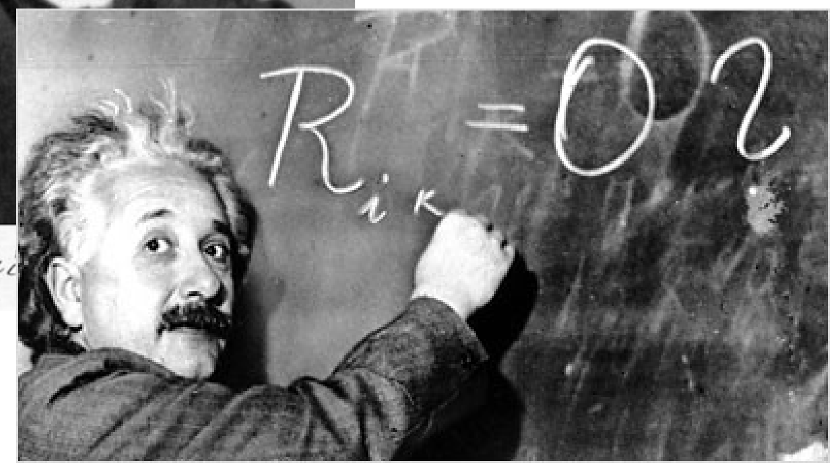




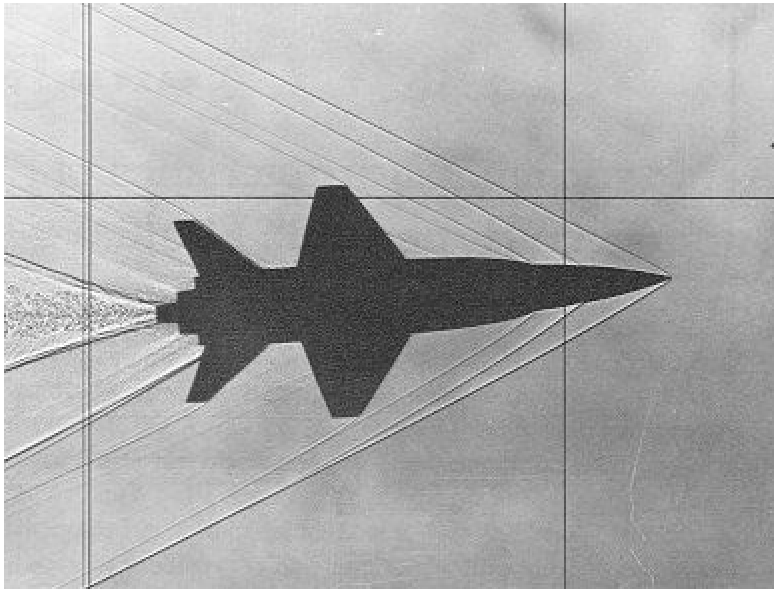


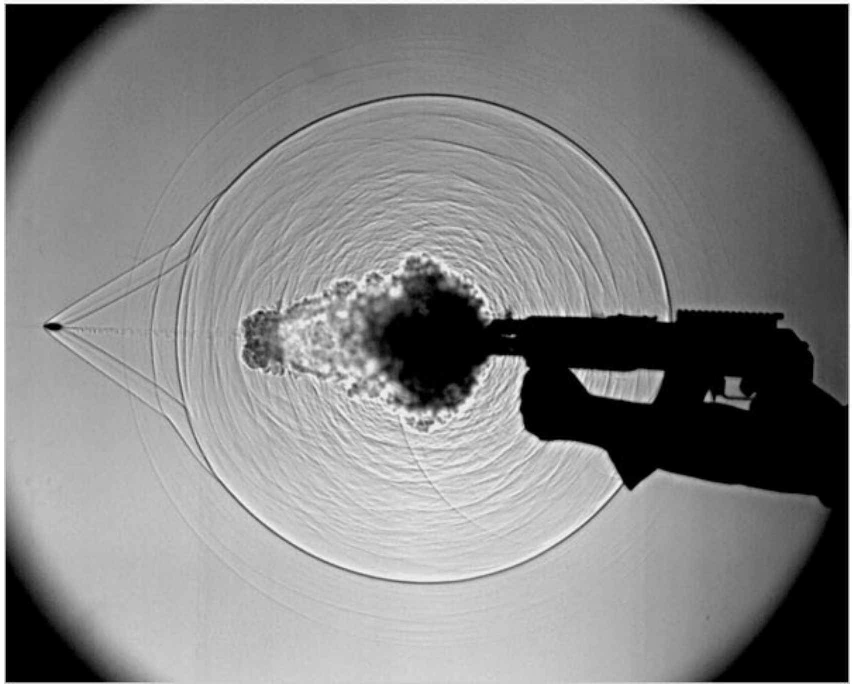
Hyperbolic surfaces





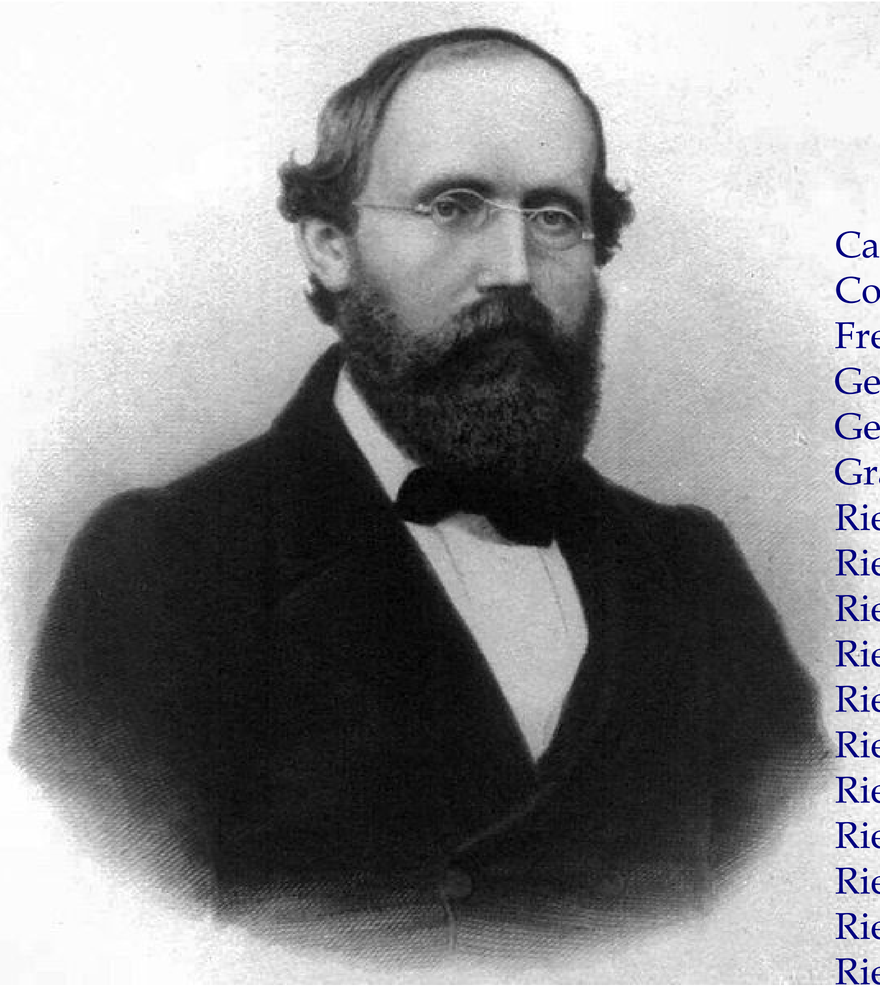
Bernhard Riemann 1826-1866





List of topics named after Bernhard Riemann

From Wikipedia, the free encyclopedia

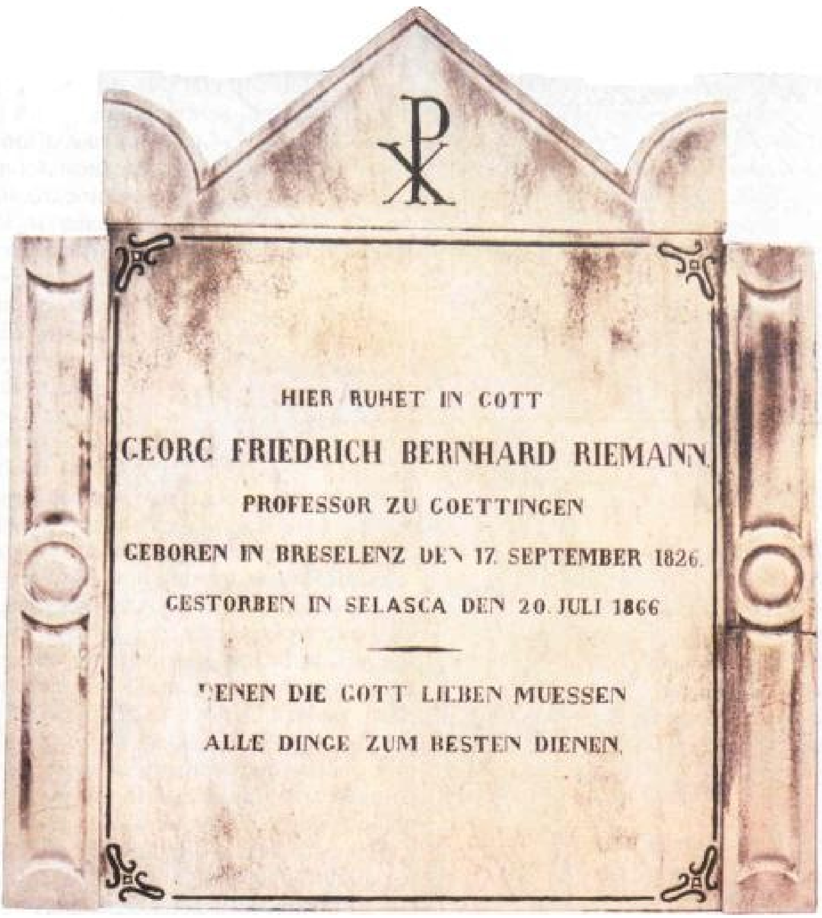


Bernhard Riemann 1826-1866

Cauchy–Riemann equations
Compact Riemann surface
Free Riemann gas
Generalized Riemann hypothesis
Generalized Riemann integral
Grand Riemann hypothesis
Riemann bilinear relations
Riemann–Cartan geometry
Riemann conditions
Riemann curvature tensor
Riemann form
Riemann function
Riemann–Hilbert correspondence
Riemann–Hilbert problem
Riemann–Hurwitz formula
Riemann hypothesis
Riemann hypothesis for finite fields
Riemann integral
Riemann–Lebesgue lemma
Riemann–Liouville differintegral
Riemann mapping theorem
Riemann matrix
Riemann multiple integral
Riemann operator
Riemann problem
Riemann–Roch theorem
Riemann series theorem
Riemann–Siegel formula
Riemann singularity theorem
Riemann solver
Riemann sphere
Riemann–Stieltjes integral
Riemann sum
Riemann surface
Riemann theta function
Riemann–von Mangoldt formula
Riemann Xi function
Riemann zeta function
Zariski–Riemann space
Riemannian bundle metric
Riemannian circle
Riemannian cobordism
Riemannian connection
Riemannian cubic polynomials
Riemannian foliation
Riemannian geometry
Riemannian graph
Riemannian group
Riemannian holonomy
Riemann invariant
Riemannian manifold
Riemannian metric tensor
Riemannian polyhedron
Riemannian decomposition
Riemannian submanifold
Riemannian submersion
Riemannian volume form

List of topics named after Bernhard Riemann

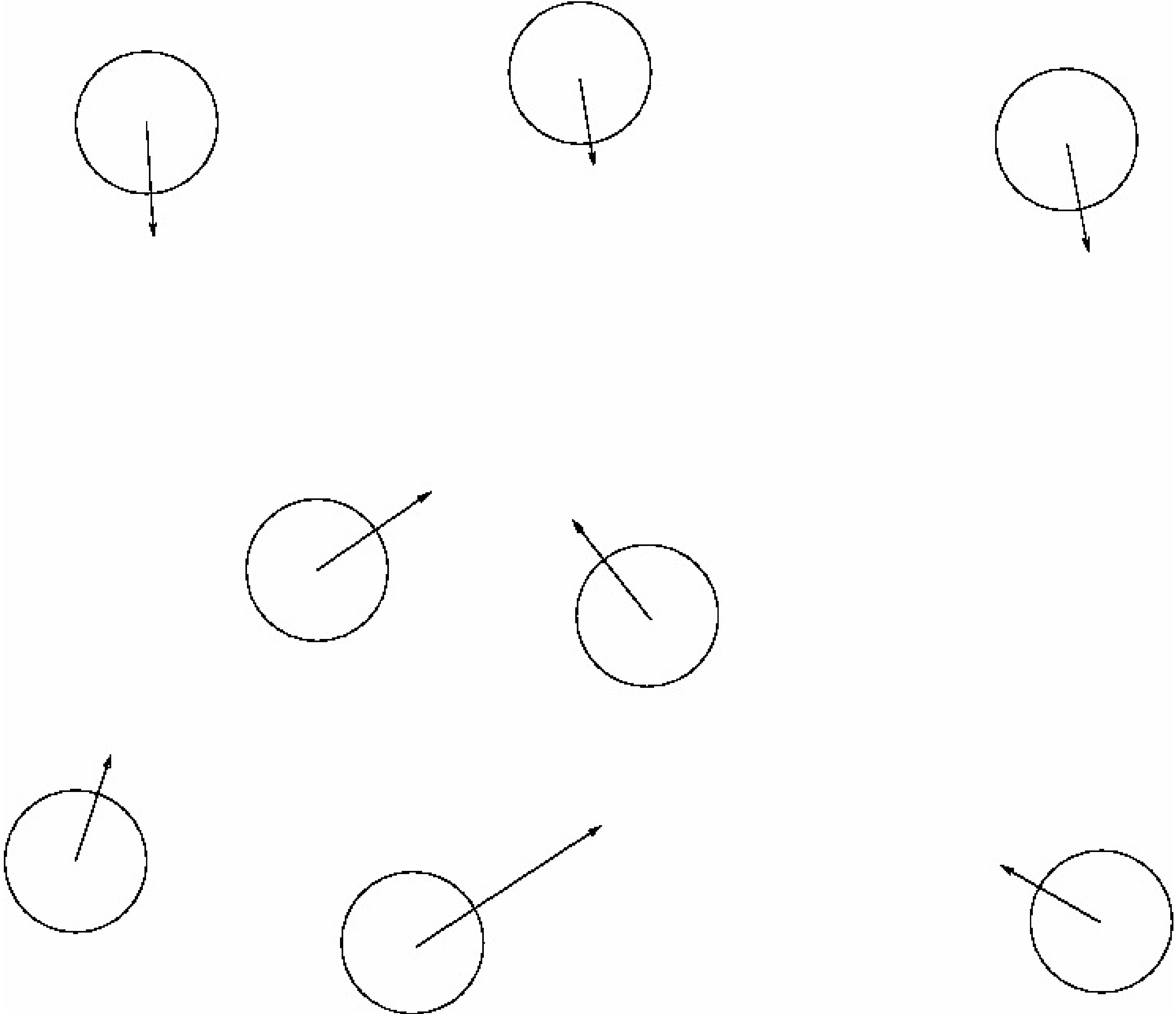
From Wikipedia, the free encyclopedia

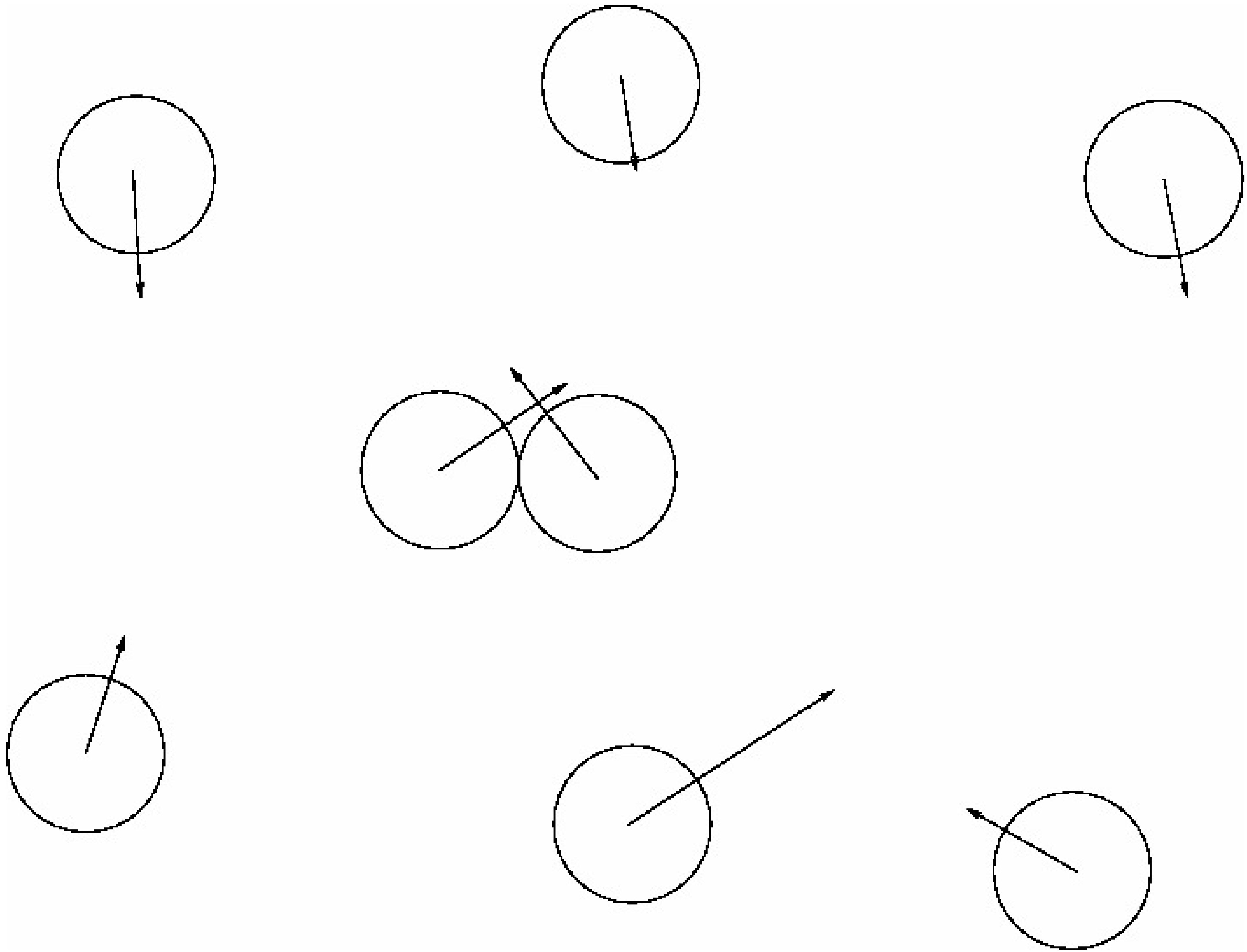


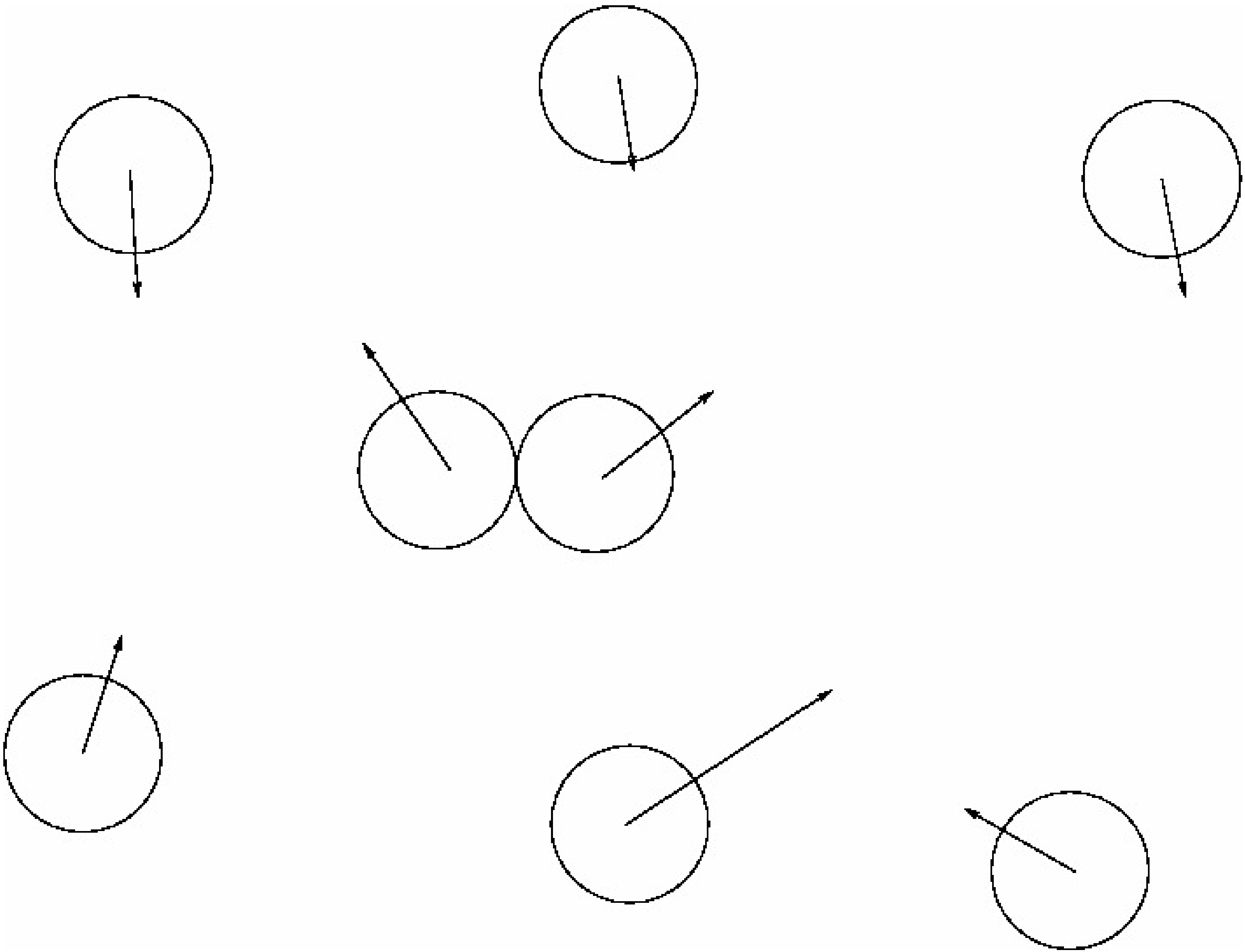
Bernhard Riemann 1826-1866

Cauchy–Riemann equations
Compact Riemann surface
Free Riemann gas
Generalized Riemann hypothesis
Generalized Riemann integral
Grand Riemann hypothesis
Riemann bilinear relations
Riemann–Cartan geometry
Riemann conditions
Riemann curvature tensor
Riemann form
Riemann function
Riemann–Hilbert correspondence
Riemann–Hilbert problem
Riemann–Hurwitz formula
Riemann hypothesis
Riemann hypothesis for finite fields
Riemann integral
Riemann–Lebesgue lemma
Riemann–Liouville differintegral
Riemann mapping theorem
Riemann matrix
Riemann multiple integral
Riemann operator
Riemann problem
Riemann–Roch theorem
Riemann series theorem
Riemann–Siegel formula
Riemann singularity theorem
Riemann solver
Riemann sphere
Riemann–Stieltjes integral
Riemann sum
Riemann surface
Riemann theta function
Riemann–von Mangoldt formula
Riemann Xi function
Riemann zeta function
Zariski–Riemann space
Riemannian bundle metric
Riemannian circle
Riemannian cobordism
Riemannian connection
Riemannian cubic polynomials
Riemannian foliation
Riemannian geometry
Riemannian graph
Riemannian group
Riemannian holonomy
Riemann invariant
Riemannian manifold
Riemannian metric tensor
Riemannian polyhedron
Riemannian decomposition
Riemannian submanifold
Riemannian submersion
Riemannian volume form

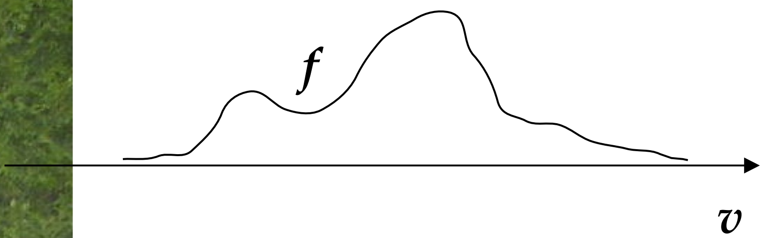
Ludwig Boltzmann 1844-1906

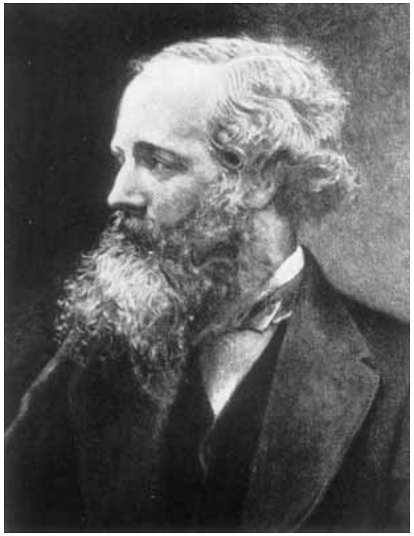






Ludwig Boltzmann 1844-1906

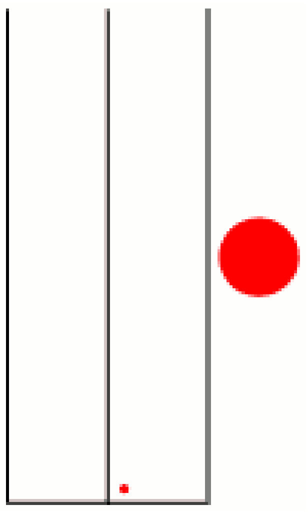


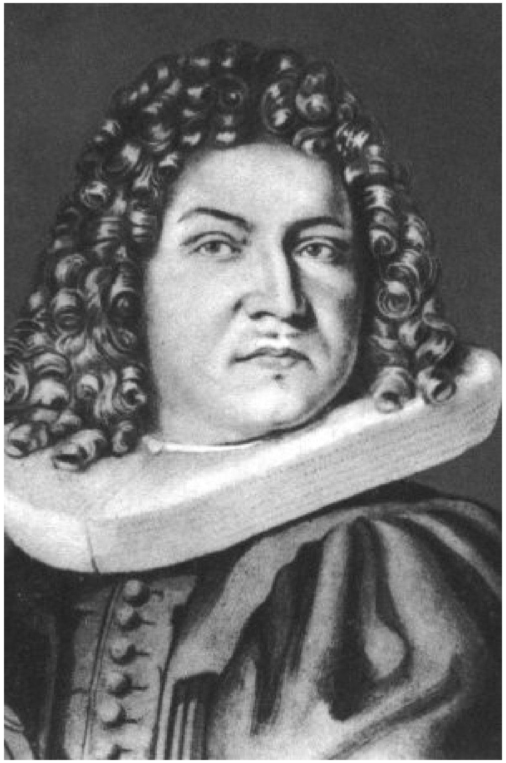


The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Law of large numbers

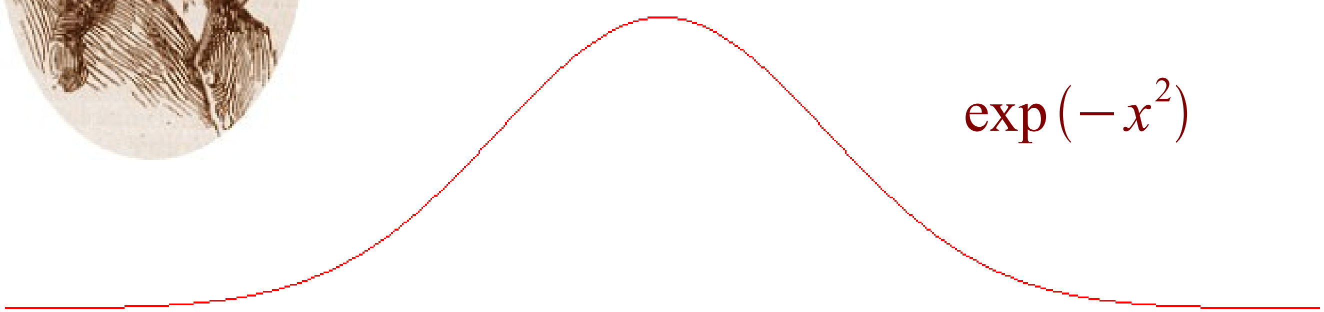




The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Abraham **de Moivre** (~ 1730) : the « Gaussian » law



$\exp(-x^2)$

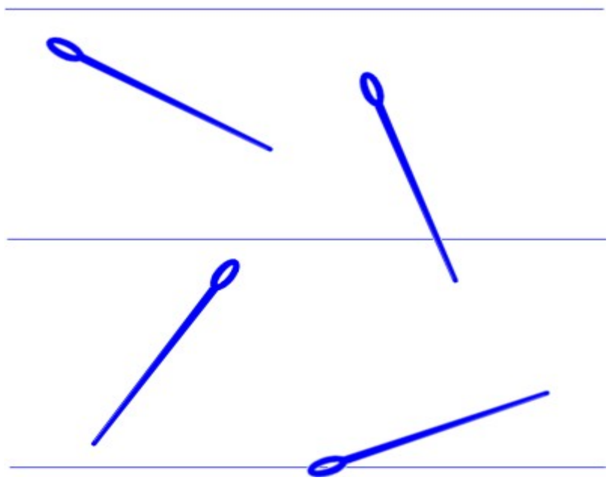
The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Abraham **de Moivre** (~ 1730) : the « Gaussian » law

Georges-Louis Leclerc de **Buffon** (1733, 1777) : the needle!

$$p = \frac{2}{\pi}$$



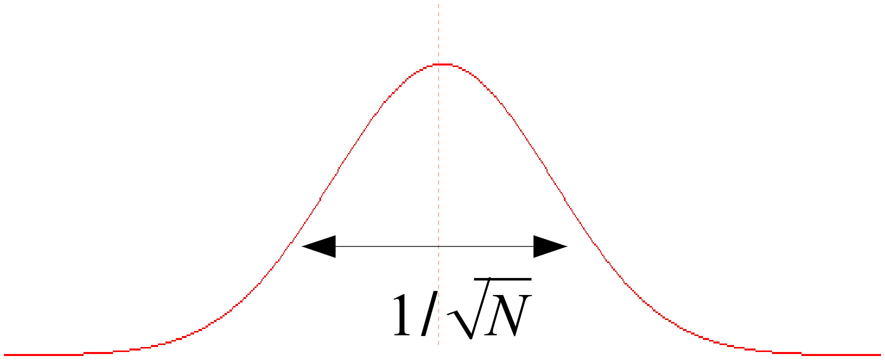
The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Abraham **de Moivre** (~ 1730) : the « Gaussian » law

Georges-Louis Leclerc de **Buffon** (1733, 1777) : the needle!

Pierre-Simon de **Laplace** (1810) : the Law of Errors



The statistical mean of a large number of random uncorrelated experiments exhibits « Gaussian » fluctuations

The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Abraham **de Moivre** (~ 1730) : the « Gaussian » law

Georges-Louis Leclerc de **Buffon** (1733, 1777) : the needle!

Pierre-Simon de **Laplace** (1810) : the Law of Errors

Adolphe **Quetelet** (1846) : makes the «law of random causes » popular

The slow mastering of Chance

Jacques **Bernoulli** (~ 1700) : statistics

Abraham **de Moivre** (~ 1730) : the « Gaussian » law

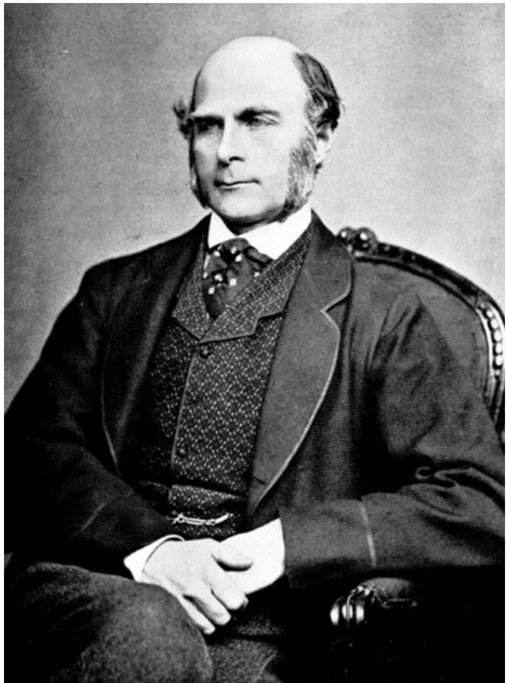
Georges-Louis Leclerc de **Buffon** (1733, 1777) : the needle!

Pierre-Simon de **Laplace** (1810) : the Law of Errors

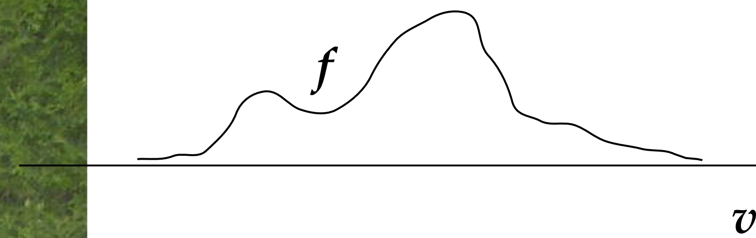
Adolphe **Quetelet** (1846) : makes the «law of random causes » popular

Francis **Galton** (1889) : « supreme law of unreason »

I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement,amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshaled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.



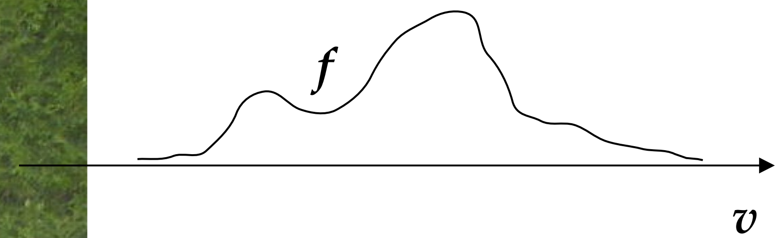
Ludwig Boltzmann 1844-1906



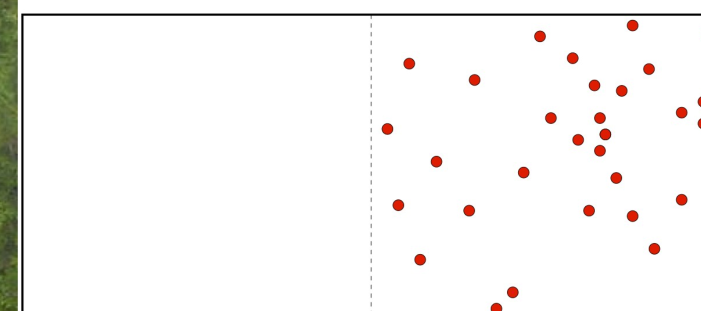
S = k log W

$$-\int_{\Omega_x \times \mathbb{R}^3_v} f(x,v) \log f(x,v) \, dv \, dx$$

Ludwig Boltzmann 1844-1906

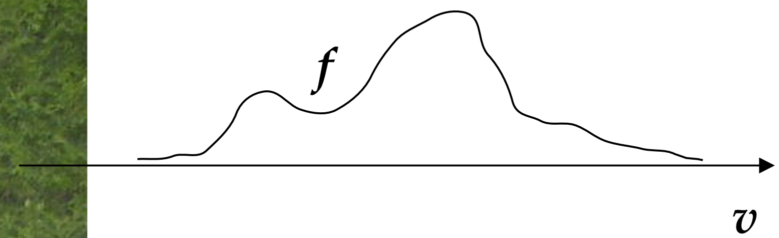


S = k log W

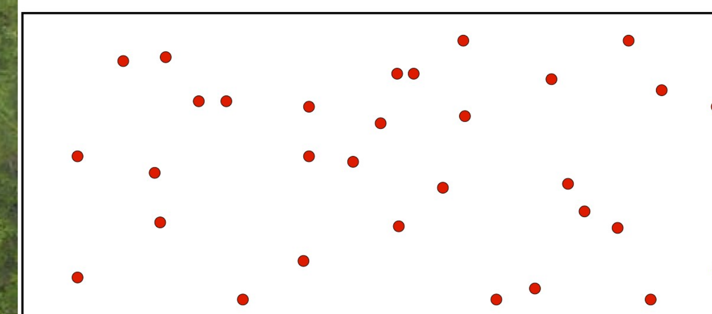


$$-\int_{\Omega_x \times \mathbb{R}^3_v} f(x,v) \log f(x,v)\, dv\, dx$$

Ludwig Boltzmann 1844-1906



S = k log W



$$-\int_{\Omega_x \times \mathbb{R}^3_v} f(x,v) \log f(x,v)\, dv\, dx$$

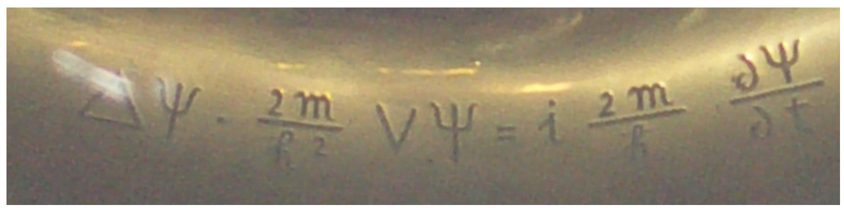
Partial Differential Equations

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \iint |v - v_*| \left[f(v') f(v'_*) - f(v) f(v_*) \right] dv_* \, d\sigma$$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V *_x \left(\int f \, dv \right) \cdot \nabla_v f = 0$$

$$\frac{\partial g}{\partial t} + 2 \, \mathrm{Ric}_g = 0$$

$$G_{\mu\nu} = 8\pi \, T_{\mu\nu}$$



$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta \rho$$

$$\partial_t u = d_u^2 \, \nabla^2 u + f(u) - \sigma v$$
$$\tau \partial_t v = d_v^2 \, \nabla^2 v + u - v$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}) = -\vec{\nabla} p + \vec{\nabla} \cdot \vec{\vec{\tau}} + \rho \vec{f}$$

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot [\, (\rho e + p) \, \vec{v} \,] = \vec{\nabla} \cdot \left(\vec{\vec{\tau}} \cdot \vec{v} \right) + \rho \vec{f} \cdot \vec{v} - \vec{\nabla} \cdot \vec{q} + r$$

$$m_i \frac{d^2 x_i}{d t^2} = -\sum_{j \neq i} G \, m_i m_j \frac{x_i - x_j}{\| x_i - x_j \|^{\beta}}$$

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

Qualitative behavior of a classical gas

ENTROPY GOES UP!

Qualitative behavior of a classical gas

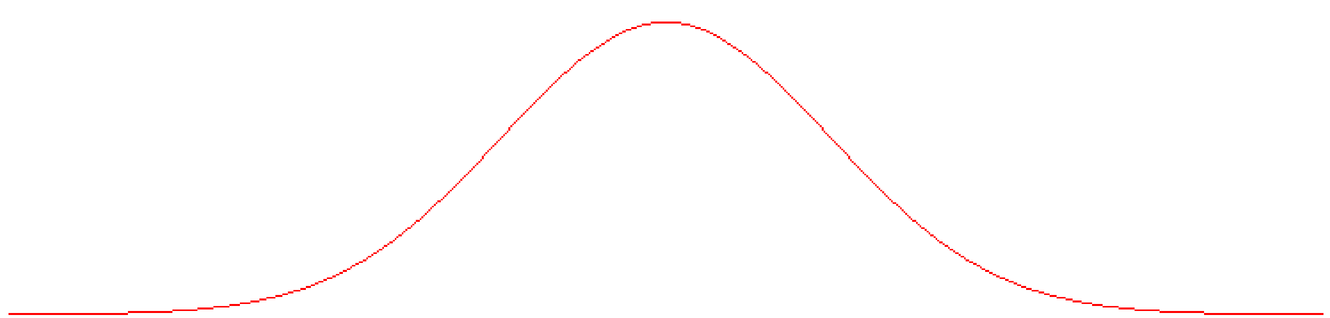
ENTROPY GOES UP!

E.g. For a homogeneous gas

$$\dot{S} = \frac{1}{4}\int \Big(f(v)f(v_*) - f(v')f(v'_*) \Big) \log \frac{f(v)f(v_*)}{f(v')f(v'_*)} \, B \, d\omega \, dv \, dv_* \quad \geq 0$$

The statistical distribution goes for a maximum entropy state

→ Gaussian distribution (again!)



Cercignani Conjecture

$$\dot{S} = \frac{1}{4}\int\Big(f(v)f(v_*) - f(v')f(v'_*)\Big)\log\frac{f(v)f(v_*)}{f(v')f(v'_*)}\,B\,d\omega\,dv\,dv_*$$

$$\geq K\,[\,S(\gamma) - S(f)\,]$$

Toscani, Villani ....

Many works on the themes

« How fast does entropy increase? »

« How fast does the gas become Gaussian ? »

Ludwig Boltzmann 1844-1906

The many faces of Entropy

Fundamental in compressible fluid mechanics (Lax...)

Key element of Shannon's Information Theory (with Fisher information)

Allows for quantitative laws of errors

Instrumental in Nash's regularity of nonsmooth diffusion equations

Allowed Perelman to prove the Poincaré conjecture

Used by Varadhan, Yau etc. to establish hydrodynamic limits

Served Voiculescu's classification of « II_1 factors »

Basis of the heat equation in metric-measure spaces (… AGS theory)

Leonid KANTOROVICH

1912 - 1986

Functional analysis; partly ordered spaces

Railroad transport, atomic bomb,
« road of life » escape, taxi fares ...

Numeric approximation, calculators...

Leonid KANTOROVICH

1912 - 1986

Functional analysis; partly ordered spaces

Railroad transport, atomic bomb,
« road of life » escape, taxi fares ...

Numeric approximation, calculators...

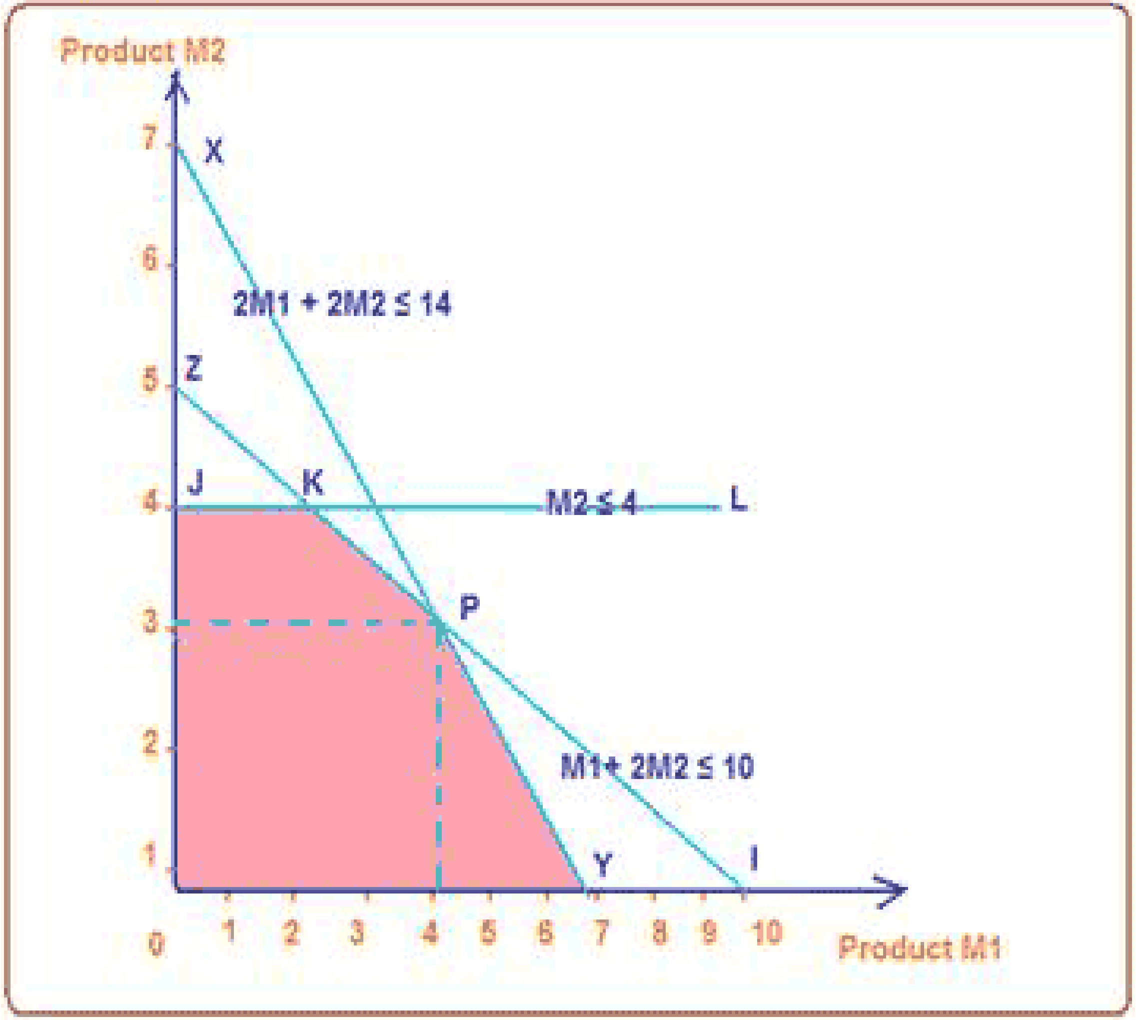
1939 : Mathematical Methods of planification and organisation of production

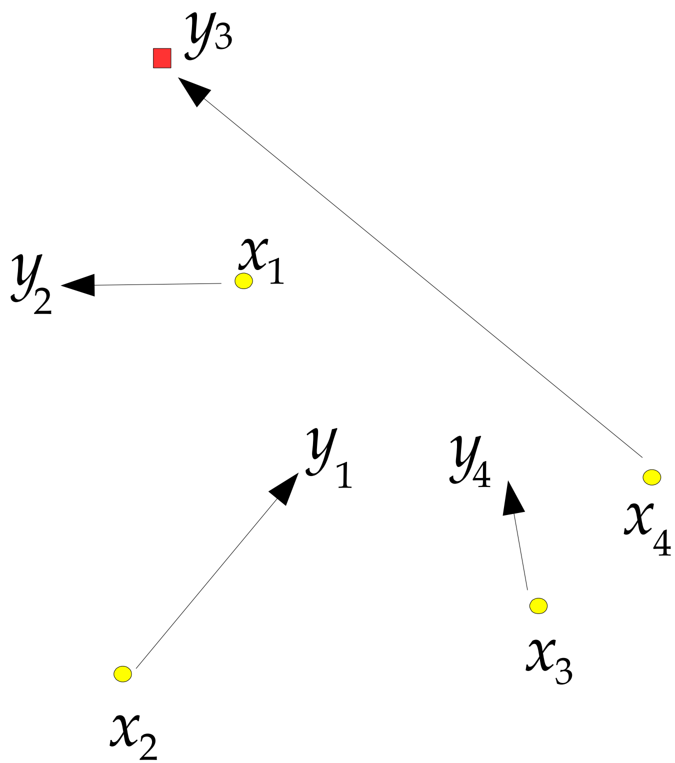
1959 : The best uses of economic resources

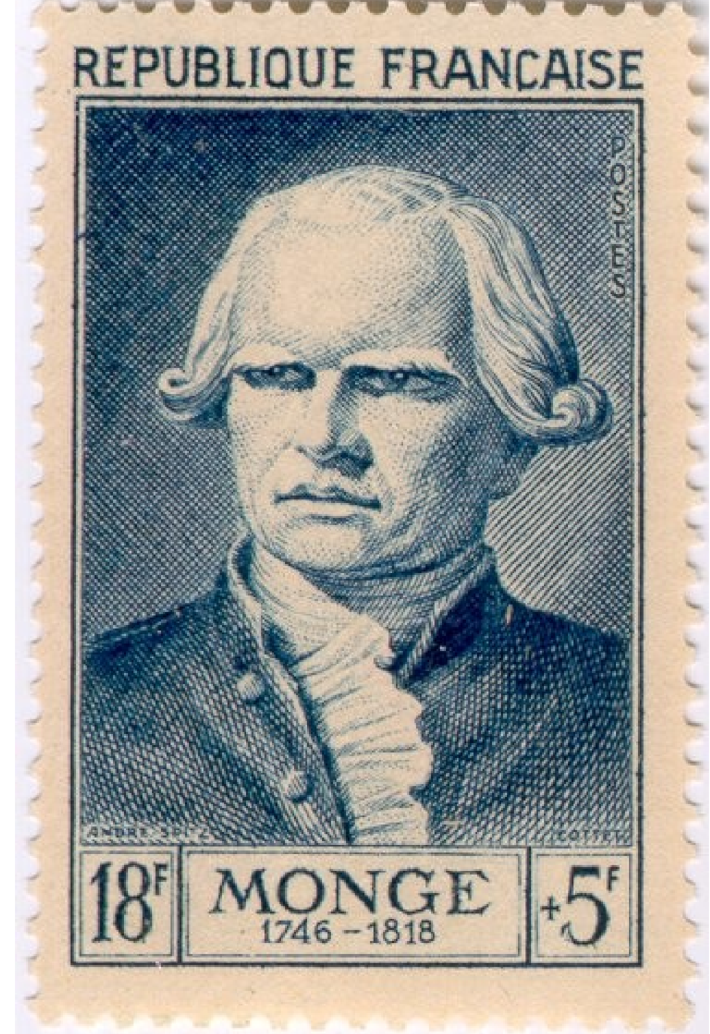
1975 : Nobel Prize in economics



Linear Programming



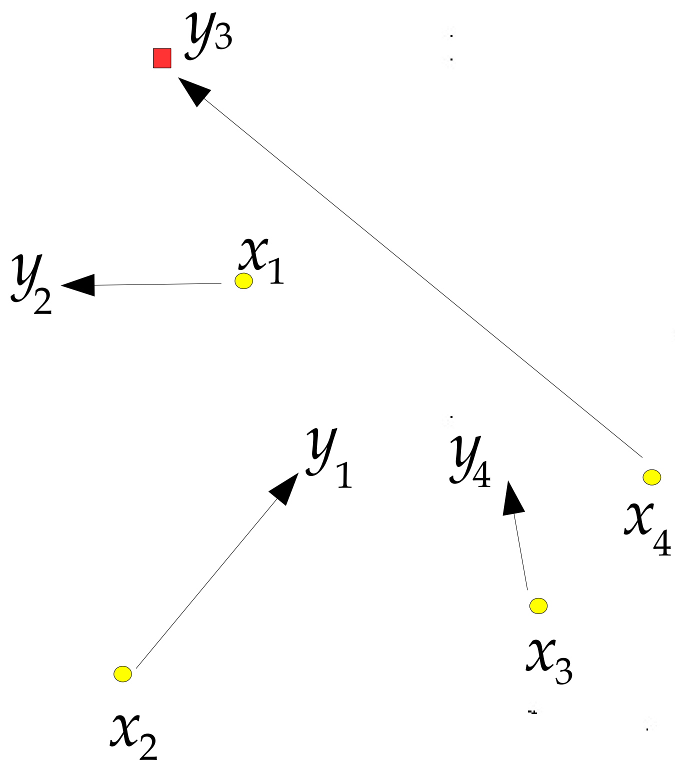


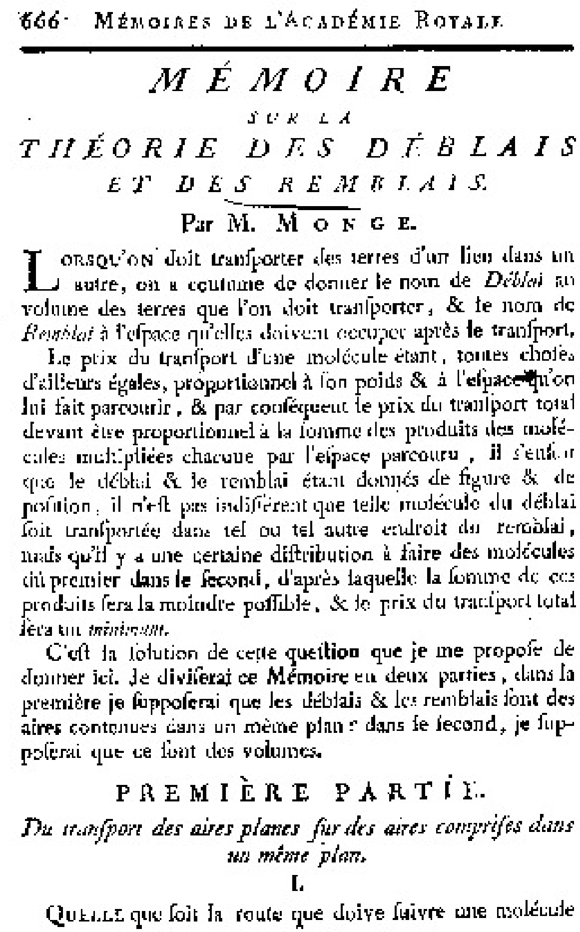


Monge-Kantorovich optimal allocation theory

$$\textbf{Min} \; \Sigma\, c(x, y) \;=\; \textbf{Max} \; \Sigma\, \phi(y) - \Sigma\, \psi(x)$$

Selling price / Buying price
– well calibrated!





666 MÉMOIRES DE L'ACADÉMIE ROYALE

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

C'eſt la ſolution de cette queſtion que je me propoſe de donner ici. Je diviſerai ce Mémoire en deux parties, dans la première je ſuppoſerai que les déblais & les remblais ſont des aires contenues dans un même plan; dans le ſecond, je ſuppoſerai que ce ſont des volumes.

PREMIÈRE PARTIE.

Du tranſport des aires planes ſur des aires compriſes dans un même plan.

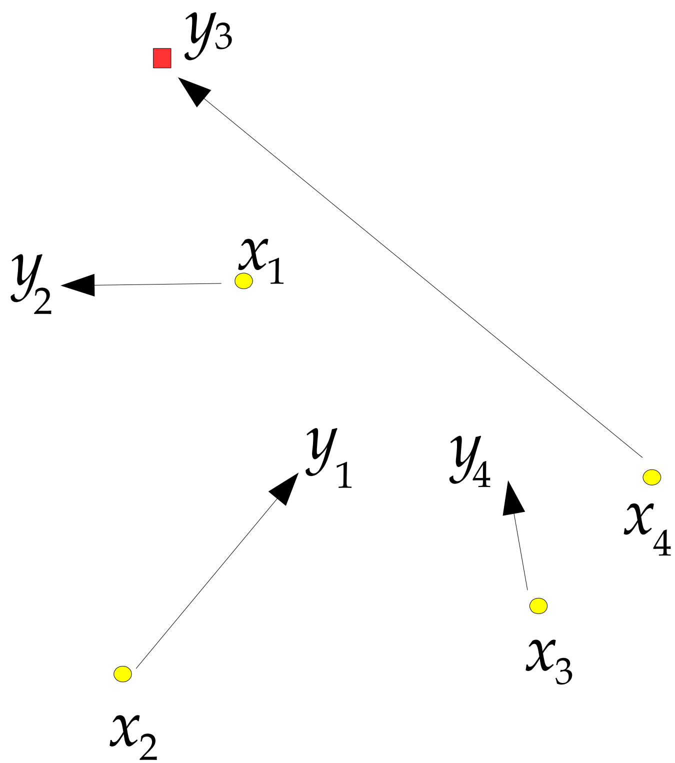
I.

QUELLE que ſoit la route que doive ſuivre une molécule

Monge-Kantorovich optimal allocation theory

$$\textbf{Min} \; \Sigma\, c(x, y) \;=\; \textbf{Max} \; \Sigma\, \phi(y) - \Sigma\, \psi(x)$$

Selling price / Buying price
– well calibrated!



Monge-Kantorovich optimal allocation theory

$$\textbf{Min} \ \Sigma\, c(x, y) = \textbf{Max} \ \Sigma\, \phi(y) - \Sigma\, \psi(x)$$

Selling price / Buying price
– well calibrated!

Product Mix Problem:
Fifth Avenue Industries

- Use 3 materials (limited resources)

Decision: How many of each type of tie to make per month?

Objective: Maximize profit

- Produce 4 types of men's ties

Media Selection Problem:
Win Big Gambling Club

Promote gambling trips t

the Bahamas

Budget: $8,000 per wee

for advertising

Use 4 types of advertisin

Linear programming

Decision: How many ads of each type?

Objective: Maximize audience reached

Blending Problem:
Whole Food Nutrition Center

Making a natural cereal that satisfies minimum daily nutritional requirements

Portfolio Selection:
International City Trust

Has $5 million to invest among 6 investments

Decision: How much to

Product Mix Problem:
Fifth Avenue Industries

- Use 3 materials (limited resources)

Decision: How many of each type of tie to make per month?

Objective: Maximize profit

- Produce 4 types of men's ties

Media Selection Problem:
Win Big Gambling Club

Promote gambling trips to the Bahamas

Budget: $8,000 per week for advertising

Use 4 types of advertising

Linear programming

Max $\sum c_i x_i$

$\sum a_{kj} x_j \leq b_k$

Decision: How many ads of each type?

Blending Problem:
Whole Food Nutrition Center

Making a natural cereal that satisfies minimum daily nutritional requirements

Portfolio Selection:
International City Trust

Has $5 million to invest among 6 investments

Objective: Maximize audience reached

Decision: How much to

Product Mix Problem:
<u>Fifth Avenue Industries</u>

- Use 3 materials (limited resources)

Decision: How many of each type of tie to make per month?

Objective: Maximize profit

- Produce 4 types of men's ties

Media Selection Problem:
<u>Win Big Gambling Club</u>

Promote gambling trips t[o] the Bahamas

Budget: $8,000 per wee[k] for advertising

Use 4 types of advertisi[ng]

Decision: How many ads of each type?

Objective: Maximize audience reached

Linear programming

AIMMS CGAL COIN-OR
CLP CVXMod Glpk
GIPALS HOPDM LINDO
LP_Solve What's Best!
PremiumSolver MOSEK
DecisionPro LP
Simplex Method Tool
Matlab Mathematica
Optimj Orstat2000 IMSL
QSOpt R SDPT3 SeDuMi ...

$$\textbf{Max} \sum c_i x_i$$

$$\sum a_{kj} x_j \le b_k$$

Blending Problem:
<u>Whole Food Nutrition Center</u>

Making a natural cereal that satisfies minimum daily nutritional requirements

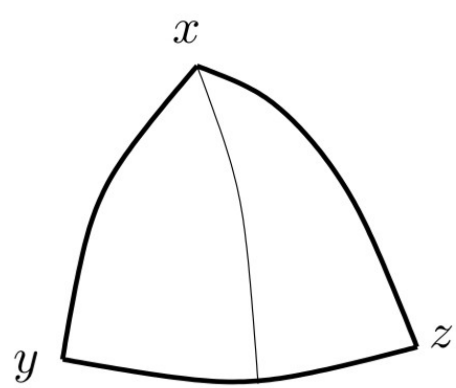
Portfolio Selection:
<u>International City Trust</u>

Has $5 million to invest among 6 investments

Decision: How much to

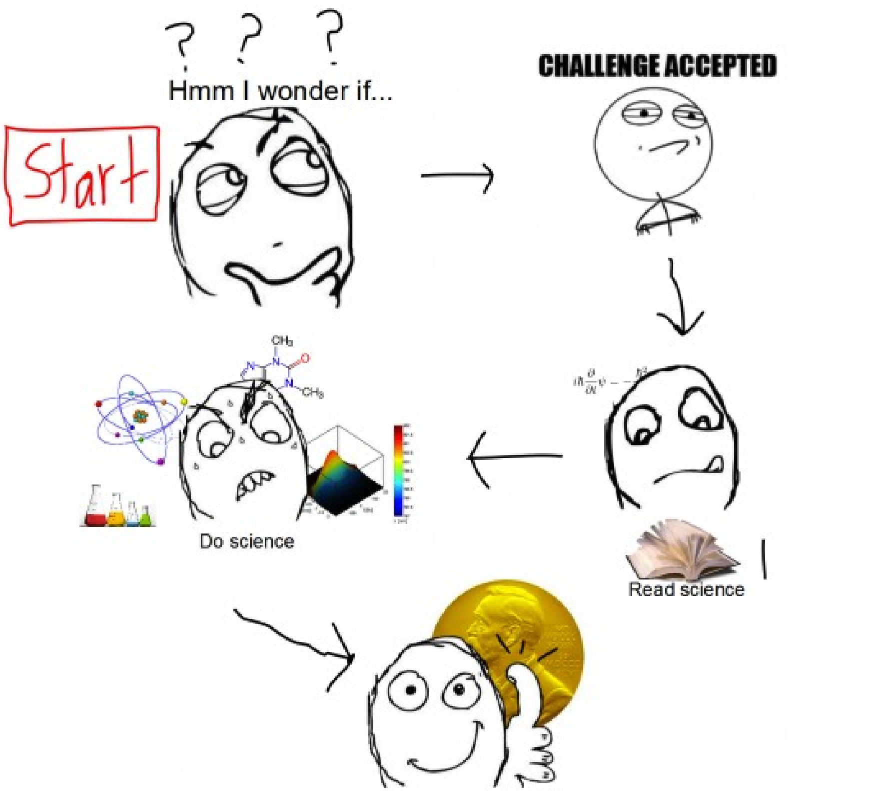


Min $\Sigma c(x, y)$

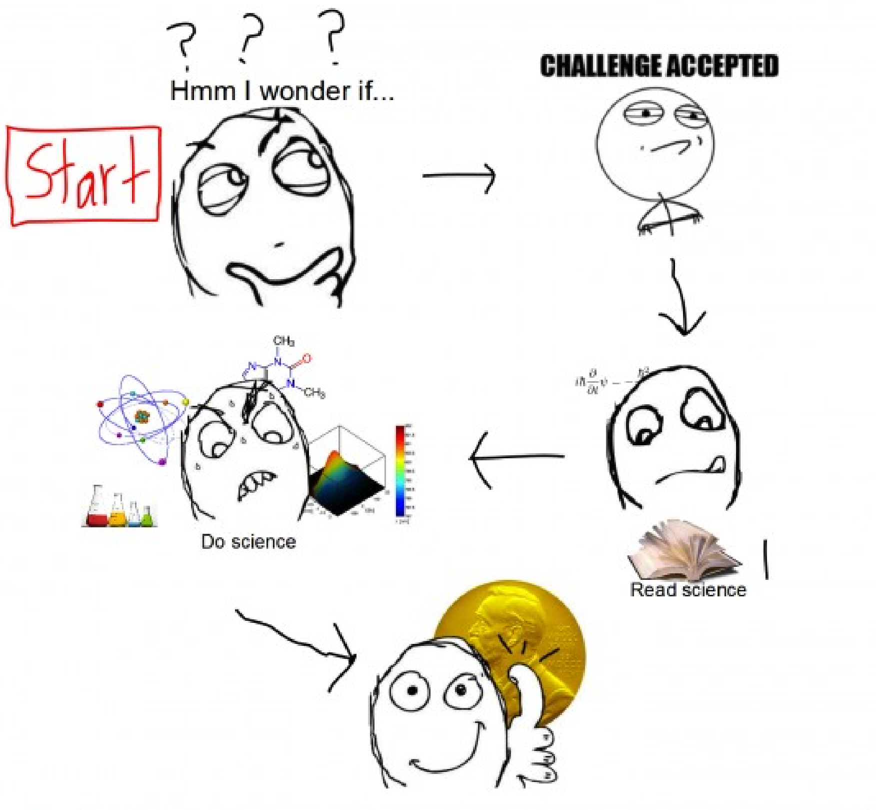


$S = -\int \varrho \log \varrho$

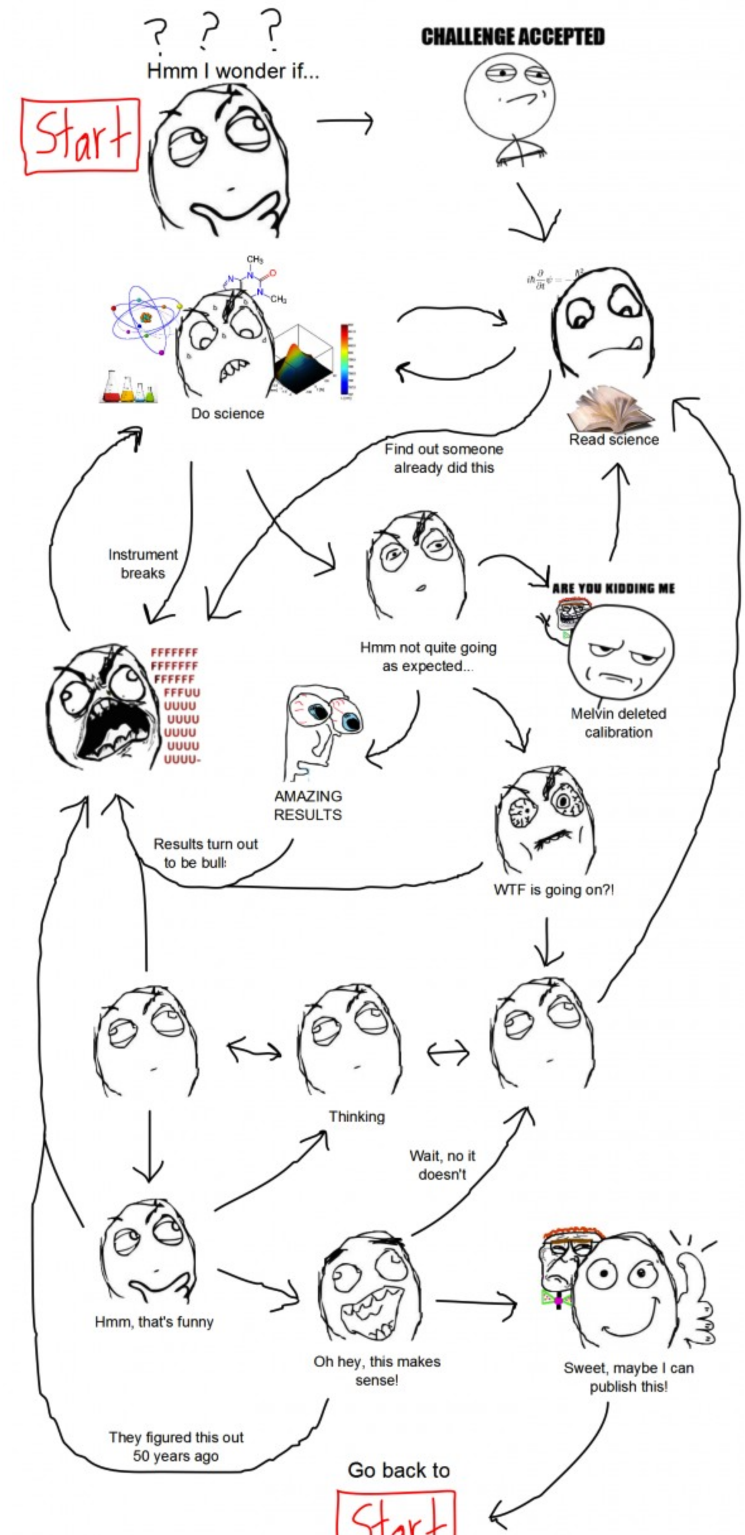
Public Perception of Science



Public Perception of Science

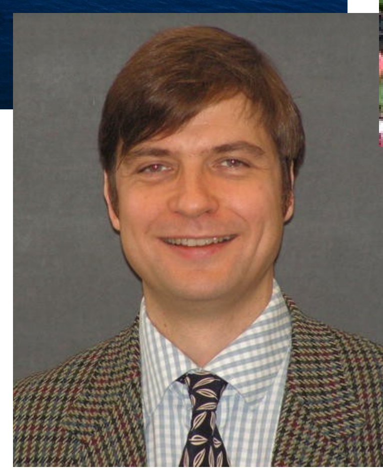


Science in Reality



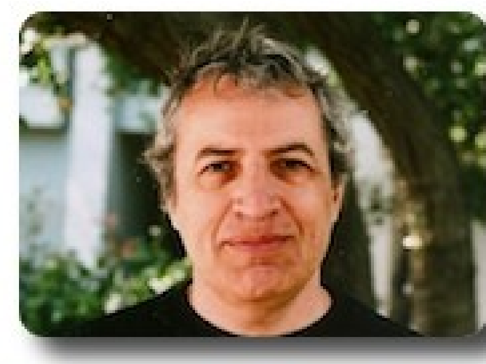
Santa Barbara (1999)

Felix Otto



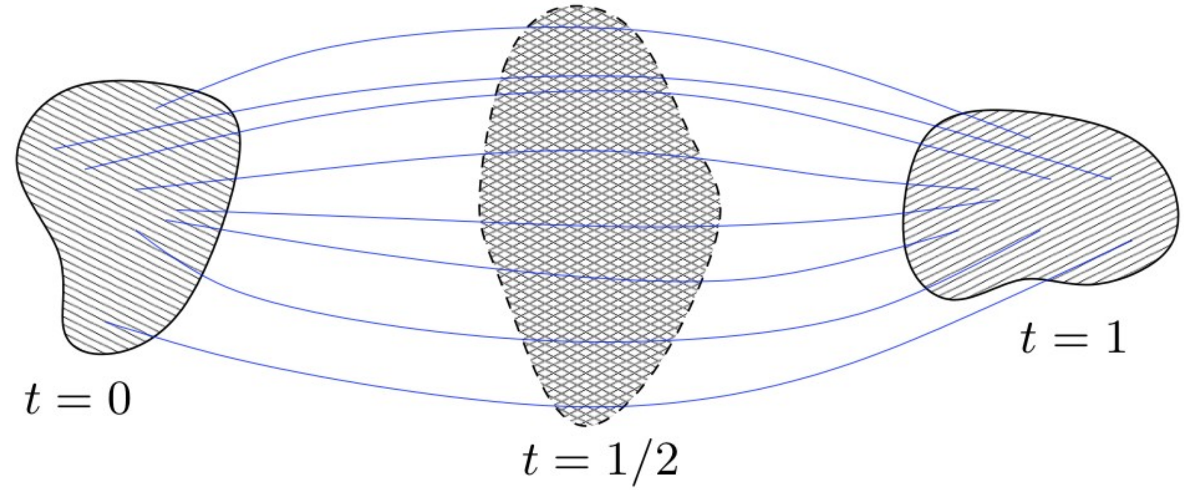
Berkeley (2004)

John Lott

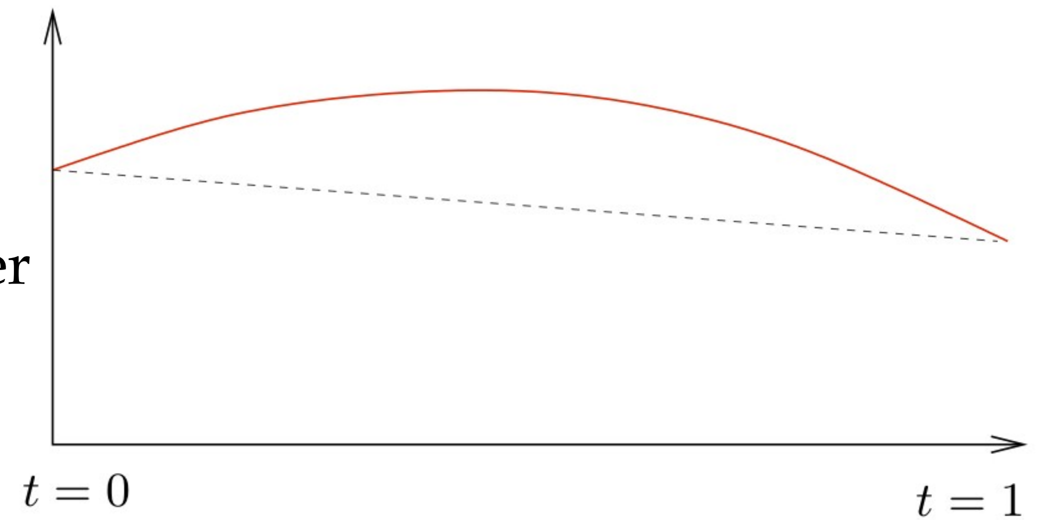


The lazy gas experiment

New way to conceive positive curvature

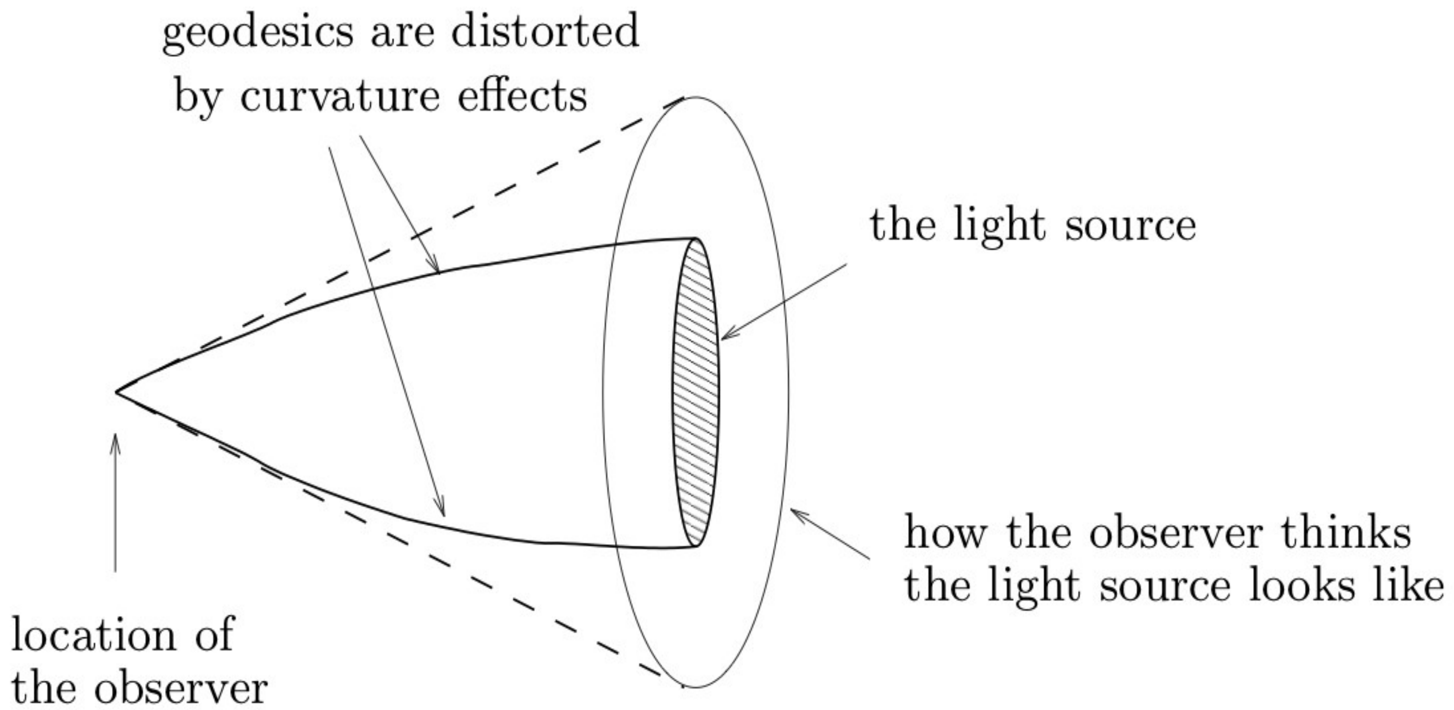


$$S = -\int \rho \log \rho$$



Otto-Villani
Cordero-McCann-Schmuckenschläger
Lott-Sturm-Villani

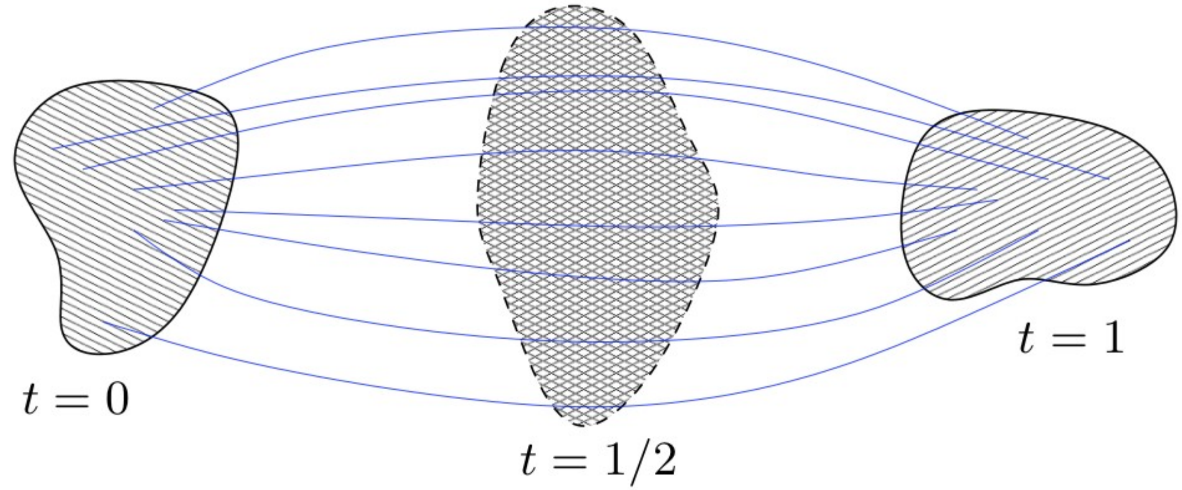
Classical « optical » interpretation of curvature



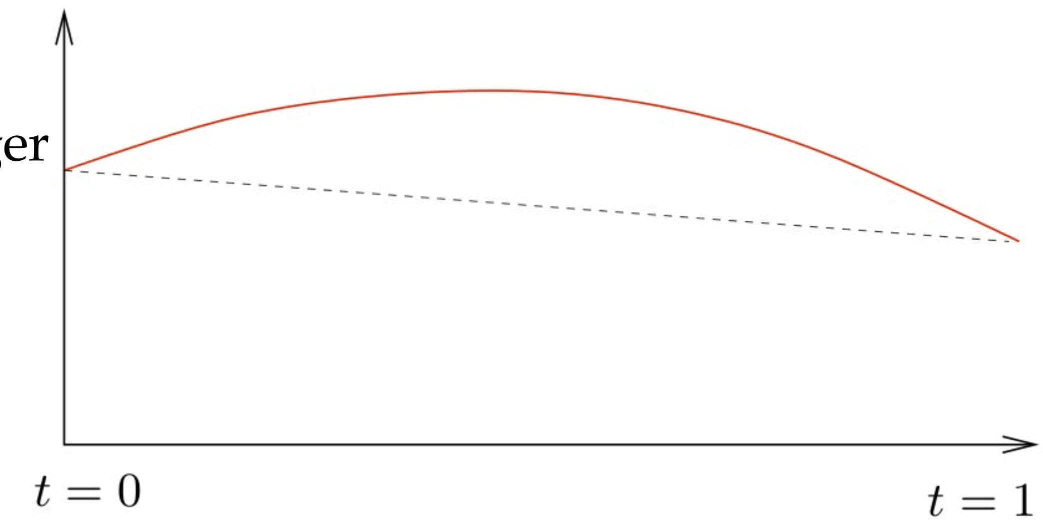
L'expérience du gaz paresseux

The lazy gas experiment

New way to conceive positive curvature



$$S = -\int \rho \log \rho$$



Otto-Villani
Cordero-McCann-Schmuckenschläger
Lott-Sturm-Villani

Optimal transport, old and new
(Springer 2008)

ihp
Institut
Henri
Poincaré