The complexity of society

How to build stable relationships between people who lie and cheat

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Building society from individuals

There are two approaches to the social sciences, depending on whether one considers the individual or the society as the starting point. Modern economic theory takes the first approach (methodological individualism), and then has to explain how individuals behave, and how they relate to each other.

Individuals are assumed to *behave rationally* (i.e. to maximize individual utility) and to *enter contracts* with each other. In so doing they face a fundamental constraint: asymmetry of information. This is usually split in two parts:

- adverse selection (lying): I know things about myself which I am not going to tell you
- moral hazard (cheating): when no one is looking, I can do whatever I want

A simple example of moral hazard

I am going to describe the original model of Sannikov (moral hazard in continuous time)

- Sannikov, "*A continuous-time version of the principal-agent problem*", RES (2008) 75, 957-984
- Sannikov, "*Contracts: the theory of dynamic principal-agent relationships and the continuous-time approach*", Working paper, 2012

Sannikov uses the PDE approach (HJB). Cvitanic uses the stochastic maximum principle approach (BSDE):

- Cvitanic and Zhang, "*Contract theory in continuous-time models*", Springer, 2012

Working for someone else

The agent is in charge of a project which generates a stream of revenue for the principal :

$$dX_t = A_t dt + \sigma dZ_t$$

where Z_t is BM, $\sigma > 0$ is given, and $A_t \geq 0$ is the agent's effort . The intertemporal utilities are:

$$\text{(principal)}\ r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(dX_t - C_t dt\right)\right]$$

$$\text{(agent)}\quad r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(u\left(C_t\right) - h\left(A_t\right)\right) dt\right]$$

where C_t is the agent's compensation (salary + bonuses), u her utility (concave, increasing) and $h\left(A_t\right)$ her cost of effort, (increasing, convex, $h\left(0\right) = 0$). Note that the agent is risk-adverse but not the principal.

The problems

Moral hazard: The principal observes X_t, $0 \leq t \leq T$, but not A_t . So C_t is conditional on $X_t, 0 \leq t \leq T$, not A_t (if C_t is Markovian, this means that $C_t = f(X_t)$)

Limited liability: The principal can reward the agent, but cannot punish her. So $C_t \geq 0$.

The contract problem: What incentive scheme can the principal devise so that the agent finds it in her own interest to exert effort ?

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- $C_t = aX_t$ (sharecropping)
- $C_t = X_t - c$ (farming)

Contracts

A *contract* is a pair (C_t, A_t), with $C_t \geq 0$ and adapted to X_t (i.e. compensation is contingent on performance). C_t is verifiable and enforceable. A_t is specified in the contract, but cannot be verified. The contract is formally for ever, but in fact can be terminated by either party:

- by setting $C_t = 0$ for $t \geq T$ the principal fires the agent

A contract (C_t, A_t) is *incentive-compatible* if the agent finds it in her own interest to exert the contractual effort A_t at every t. It is *individually rational* if both the principal and the agent find it in their own interest to enter the contract at $t = 0$.

$$\text{(principal)} \quad r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(dX_t - C_t dt\right)\right] \geq 0$$

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- by setting $C_t = c > 0$ for $t \geq T$ the principal pensions off the agent

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Finding incentive-compatible contracts:

We look at the agent's *continuation value*

$$\begin{aligned} W_t &= r\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\left(u\left(C_s\right)-h\left(A_s\right)\right)ds \mid \mathcal{F}_t^Z\right] \\ &= re^{rt}\mathbb{E}\left[\int_0^\infty e^{-rs}\left(u\left(C_s\right)-h\left(A_s\right)\right)ds \mid \mathcal{F}_t^Z\right] \\ &\quad -re^{rt}\int_0^t e^{-rs}\left(u\left(C_s\right)-h\left(A_s\right)\right)ds \end{aligned}$$

The first term is a martingale. Using the martingale representation theorem we find that there is a Z_t-adapted process Y_t (depending on C_t and A_t) such that:

$$\begin{aligned} \frac{1}{r}dW_t &= \left(W_t - u\left(C_t\right)+h\left(A_t\right)\right)dt + Y_t\sigma dZ_t \\ &= \left(W_t - u\left(C_t\right)+h\left(A_t\right)-Y_tA_t\right)dt + Y_t dX_t \end{aligned}$$

There is a *fixed part* and a *variable part* depending on performance.

Incentive compatible contracts

Suppose the agent has conformed to the contract (C_s, A_s) for $s \leq t$, and tries to cheat, by performing effort a in the following interval $[t,\ t+dt]$, and reverting to A_s for $s \geq t + dt$

- her immediate compensation $C_t dt$ is unaffected
- her cost on $[t,\ t+dt]$ is $rh(a)\,dt$
- her expected benefit on $[0,\ \infty]$ is $\mathbb{E}[Y_t dX_t] = rY_t a dt$
- the balance is $r(aY_t - h(a))\,dt$

It turns out that testing for such small deviations is enough:

Theorem (One-shot rule)

The contract (A_t, C_t) is (IC) if and only if:

$$Y_t A_t - h(A_t) = \max_{0 \leq a} \{aY_t - h(a)\} \quad a.e \tag{1}$$

A beautiful proof

Suppose (A, C) does not satisfy condition (1). Then there is an alternative contract (A^*, C^*) with:

$$\begin{array}{rcl} Y_t A_t^* - h(A_t^*) & \geq & Y_t A_t - h(A_t) \quad \text{a.e} \\ P\left[Y_t A_t^* - h(A_t^*)\right] & > & P\left[Y_t A_t - h(A_t)\right] \end{array}$$

The agent picks $t > 0$ and plans to apply A^* for $s \leq t$ and A for $\geq t$. Expected utility at t, conditional on $Z_{:t}$:

$$\begin{array}{rcl} \frac{1}{r} V_t^* & = & \mathbb{E}\left[\int_0^\infty e^{-rt}\left(u(C_t) - h_t\right) ds \mid \mathcal{F}_t^Z\right] \\ & = & \int_0^t e^{-rs}\left(u(C_s) - h(A_s^*)\right) ds + e^{-rt} W_t(A, C) \\ & = & W_0(A, C) + \int_0^t e^{-rs}\left(h(A_s) - h(A_s^*) - Y_s A_s + Y_s A_s^*\right) ds(2) \\ & & + \int_0^t e^{-rs} Y_s \left(dX_s - A_s^* ds\right) \end{array}$$

A beautiful proof, ct'd

Since $dX_s = A_s^* dt + \sigma dW_s$ for $s \leq t$, the last term is a martingale. Hence:

$$\mathbb{E}\left[V_t^*\right] = W_0\left(A, C\right) + \mathbb{E}\left[\int_0^t e^{-rs}\left(h\left(A_s\right) - h\left(A_s^*\right) - Y_s A_s + Y_s A_s^*\right) ds\right]$$

The integrand is non-negative, and positive on a set of positive measure in (t, ω). It follows that there is some $\bar{t}$ such that $\mathbb{E}\left[V_{\bar{t}}^*\right] > W_0\left(A, C\right)$. But this means that switching from A^* to A at time $\bar{t}$ is better than sticking with A from the beginning. So $\left(A, C\right)$ cannot be (IC).
Conversely, if $\left(A, C\right)$ satisfies condition (1), then, by formula (2), V_t^* is a supermartingale, with:

$$W_0\left(A, C\right) = V_0^* \geq \mathbb{E}\left[V_\infty^*\right] = W_0\left(A^*, C\right)$$

so A_t is at least as good as any alternative strategy A_t^*.

The optimal control problem

If $Y_t = h'(A_t)$, the contract is incentive-compatible, so the agent will put in effort A_t, even unobserved, and the principal can now observe $dZ_t = \sigma^{-1}(dX_t - A_t dt)$ and devise his optimal contract (C_t, A_t). Sannikov's idea consist of considering W_t as a *performance index* to be constructed along the trajectory), and on conditioning the contract on W_t:

$$\max_{C_t, A_t, T} \mathbb{E}\left[\int_0^T e^{-rt}(dX_t - C_t dt)\right]$$

$$\begin{aligned} \frac{1}{r} dW_t &= (W_t - u(C_t) + h(A_t))\, dt + h'(A_t)\, \sigma dZ_t \\ W_t &\geq 0 \quad 0 \leq t \leq T \end{aligned}$$

Mathematically speaking, the *state* is W_t, the *controls* are $C_t \geq 0$, $A_t \geq 0$, $W_0 \geq 0$ (the initial value) and $T \geq 0$ (the stopping time)

The value function

Suppose $C_t = c$ for $t \geq T$ (the principal pensions off the agent at time T). Then $A_t = 0$ for $t \geq T$ (the agent stops working). We then have $c = u^{-1}(W_T)$:

$$W_T = r \int_T^\infty e^{-r(t-T)} u(c)\, dt = u(c)$$

and the utility for the principal becomes:

$$r\mathbb{E}\left[\int_0^T e^{-rt}(dX_t - C_t dt)\right] - e^{-rT} u^{-1}(W_T)$$

The value function $F(w)$ is the highest expected utility the principal can obtain while delivering to the agent the expected utility w:

$$F(w) = \sup \mathbb{E}\left[r \int_0^T e^{-rt}(dX_t - C_t dt) - e^{-rT} u^{-1}(W_T) \mid W_0 = w\right]$$

The Hamilton-Jacobi-Bellman equation

$F : [0, \ \infty) \to R$ is continuous and $F(w) \geq -u^{-1}(w)$ everywhere. T is the first time when $F(W_t) \leq -u^{-1}(W_T)$.
$F(w)$ must satisfy a *quasi-variational inequality*:

$$F(w) = \max_{a \geq 0, c \geq 0} \left\{ \begin{array}{c} -u^{-1}(w), \\ a - c + F'(w)(w - u(c) + h(a)) + \frac{r}{2} F''(w) h'(a)^2 \sigma^2 \end{array} \right.$$

The optimal controls are given by:

$$a_{\max} = A(w) \quad \text{and} \quad c_{\max} = C(w)$$

The verification theorem.

There is an optimal contract, which is Markovian w.r.t W_t:

Theorem

Suppose F solves (IQV) with $F(0) = 0$. Pick some w and define W_t as follows:

$$\frac{1}{r} dW_t = W_t - u\left(C\left(W_t\right) + h\left(A\left(W\right)\right) - h'\left(A\left(W_t\right)\right) A\left(W_t\right)\right) dt + h'(A(W_t \tag{3}$$

with $W_0 = w$ Then the contract $C_t = C(W_t)$, $A_t = A(W_t)$ is (IC), (IR), and has value w for the agent and $F(w)$ for the principal. The principal fires the agent when $W_T = 0$ and retires him when $F(W_t) = -u^{-1}(W_t)$. Any (IC) (IR) contract starting from $W_0 = w$ yields to the principal a profit less than or equal to $F(w)$

The principal then chooses $w = w_0 = \arg\max_w F(w)$

Proof: computing the value for the principal

By construction, the continuation value of that contract for the agent is W_t. It is (IC) and (IR) by construction. Its value to the principal is:

$$\left[\int_0^t re^{-rs}\left(A_s - C_s ds\right) - e^{-rt}u^{-1}\left(W_T\right)\right]$$

The random variable $G_{t:} := \int_0^t re^{-rs}\left(A_s - C_s ds\right) + e^{-rt}F\left(W_t\right)$ is a diffusion. Applying Ito's Lemma to 3 and using HJB, we see that it is a martingale. By the optional stopping theorem,

$$\mathbb{E}\left[G_T\right] = G_0 = F\left(W_0\right)$$

One then checks that this contract is optimal, provided the solution F is convex.

Exercises

- why does the principal continue the contract even when $F(W_t) < 0$, i.e. when she expects to lose money ?
- why does the principal pension the agent instead of simply firing him ?
- why does the principal fire the agent when the index $W_t = 0$ is reached (left bound) ? Wouldn'it is be better to start off again at w_0 ?

Conclusions

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There are two remarkable facts:

- the optimal contract is Markovian, depending only on the current value of an appropriate index
- the one-shot deviation principle: the agent's incentive constraints hold for all alternative strategies A_t^* if they hold for all strategies which differ from A_t for an infinitesimally small time

Extensions

The moral hazard / limited liability complex has now been investigated in several different situations:

- How can the owners of a firm induce the managers to spend enough effort in preventing accidents, in an industry where accidents are rate but very costly (to the owners) ? See Biais, Mariotti, Rochet, Villeneuve, "*Large risks, limited liability and dynamic moral hazard*", EMA (2010), 73-118
- What kind of contracts should financiers pass with entrepreneurs, when the latter have the possibility not to exert due diligence, or even to divert investment money to other purposes ? See Biais, Mariotti, Plantin, Rochet, "*Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications*", The Review of Economic Studies (2007) p. 345-390.
- All these (second-best) contracts turn out not to be renegociation-proof . Can one devise incentive-compatible contracts which are renegociation-proof (third-best) ? Work in progress by IE and Rochet

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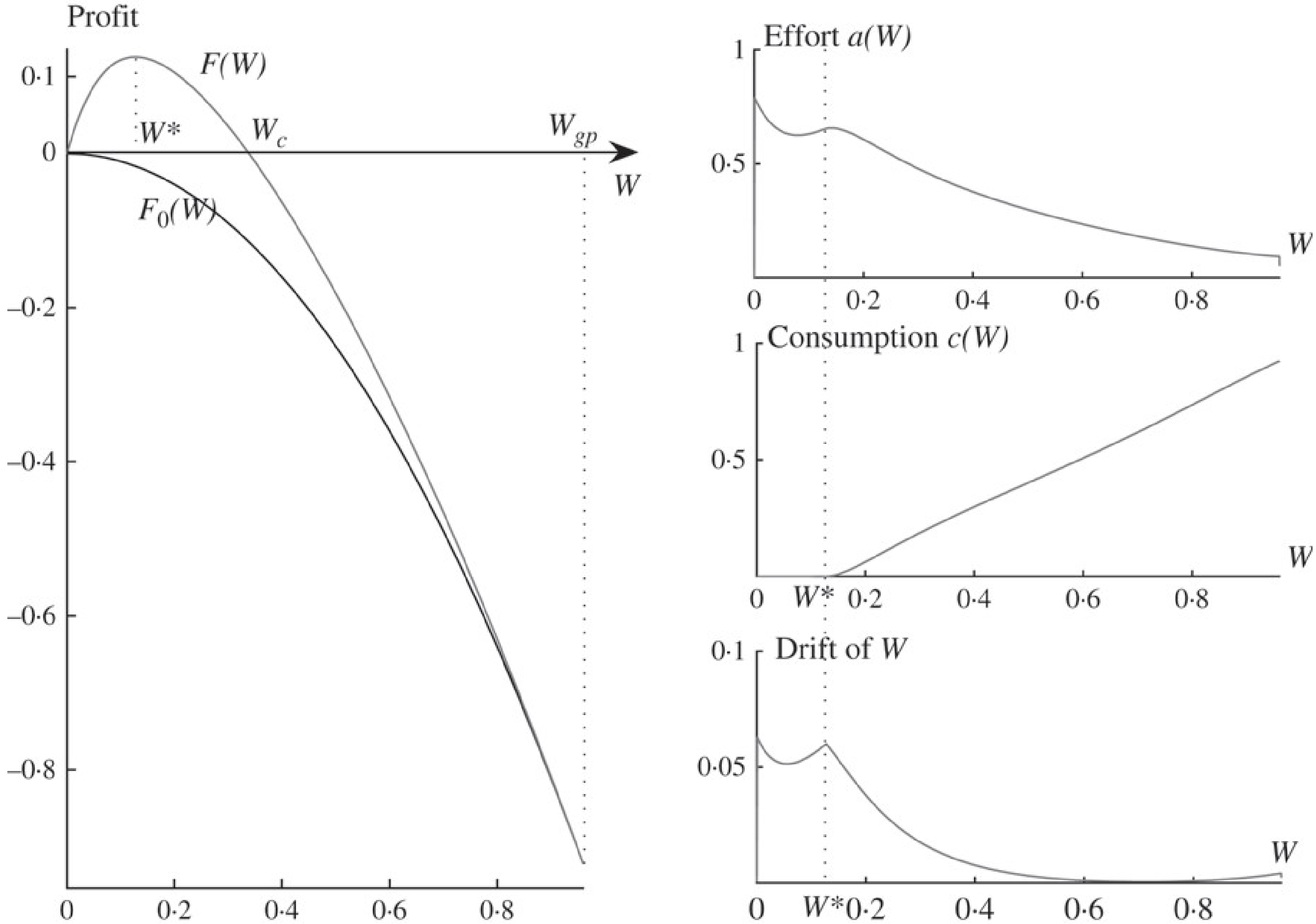
  - Hold managers accountable (prosecutions)
- One can also question the basic assumptions: individual methodology and rationality. Perhaps societies have other means than contracts to influence behaviour (education, ethics)


Profit
F(W)
W*
W_c
W_gp
W
F_0(W)
0·1
0
−0·2
−0·4
−0·6
−0·8
0
0·2
0·4
0·6
0·8
Effort a(W)
1
0·5
W
0
0·2
0·4
0·6
0·8
Consumption c(W)
1
0·5
W
0
W*
0·2
0·4
0·6
0·8
Drift of W
0·1
0·05
W
0
W*0·2
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